NAME:

Solutions to Math 150 Practice Exam 3.1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Find a simple expression for $\int \left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right) dx$ [10 pts]

Solution: $\int \left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right) dx = \int \left(4x^{1/2} - 4x^{-1/2}\right) dx = \frac{8}{3}x^{3/2} - 8x^{1/2} + C$

2. Find a simple expression for $\int \sec 4\theta \tan 4\theta \, d\theta$ [10 pts]

Solution: $\int \sec 4\theta \tan 4\theta \, d\theta = \frac{1}{4}\sec 4\theta + C$

3. Use geometry to evaluate $\int_0^4 \sqrt{16 - x^2} dx$ [10 pts]

Solution: Observe that the integral represents the quarter area of the circle with center at the origin and radius 4. To see this, let $y = \sqrt{16 - x^2}$. Then $0 \le y \le 4$ when $0 \le x \le 4$. Upon squaring, we obtain $y^2 = 16 - x^2$ or $x^2 + y^2 = 16$. Hence,

$$\int_0^4 \sqrt{16 - x^2} dx = \frac{16\pi}{4} = 4\pi$$

4. Use Riemann sums to evaluate $\int_1^4 (x^2 - 1) dx$ [10 pts]

Solution:

$$\int_1^4 (x^2 - 1) dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(1 + k\frac{3}{n}\right)^2 - 1\right]\frac{3}{n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(1 + k\frac{3}{n}\right)^2 - 1\right]\frac{3}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \left[k\frac{6}{n} + k^2\frac{9}{n^2}\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \left(\frac{18}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{18}{n^2}\frac{n(n+1)}{2} + \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right) = 18$$

5. Compute $\lim_{n\to\infty} \frac{2}{n}\left(\sqrt{1+1\frac{2}{n}}+\sqrt{1+2\frac{2}{n}}+\cdots+\sqrt{1+n\frac{2}{n}}\right)$

[10 pts]

Solution:

$$\lim_{n\to\infty} \frac{2}{n}\left(\sqrt{1+1\frac{2}{n}}+\sqrt{1+2\frac{2}{n}}+\cdots+\sqrt{1+n\frac{2}{n}}\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sqrt{1+k\frac{2}{n}}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\sqrt{1+k\frac{2}{n}} = \int_1^3 \sqrt{x}\,dx = \frac{2}{3}3^{3/2}-\frac{2}{3}1^{3/2} = 2\sqrt{3}-\frac{2}{3}$$

6. Find $\frac{d}{dx}\int_{x^2}^{10}\frac{dz}{z^2+1}$ [10 pts]

Solution: $\frac{d}{dx}\int_{x^2}^{10}\frac{dz}{z^2+1} = -\frac{1}{(x^2)^2+1}2x = -\frac{2x}{x^4+1}$

7. Find the average value of the function $f(x) = x^3$ over the interval [-1, 1]

[10 pts]

Solution: $f_{ave} = \frac{1}{2}\int_{-1}^{1} x^3\,dx = 0$

8. Calculate $\int_0^{\pi/2}\sin^2\theta\cos\theta\ d\theta$ [10 pts]

Solution: $\int_0^{\pi/2}\sin^2\theta\cos\theta\ d\theta = \frac{1}{3}\sin^3\theta\Big|_0^{\pi/2} = 1/3$

9. Find a simple expression for $\int t^3\sin t^4 dt$ [10 pts]

Solution: $\int t^3\sin t^4 dt = -\frac{1}{4}\cos t^4 + C$

10. Suppose that $\lim_{h\to 0}\frac{1}{h}\int_2^{2+h} f(t)dt = 5$. Compute $\lim_{h\to 0}\frac{1}{h}\int_2^{2-3h} f(t)dt$

[10 pts]

Solution: $\lim_{h\to 0}\frac{1}{h}\int_2^{2-3h} f(t)dt = -3\frac{1}{-3h}\int_2^{2-3h} f(t)dt = -15$

Extra-Credit

11. Under what conditions on $f(x)$ can the limit $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt$ be easily computed? Explain your answer. [10 pts]

Solution: If the integrand $f(t)$ is continuous at $t = x$, the values $f(t)$ are nearly the same as $f(x)$ for all $t \in [x, x+h]$ and all sufficiently small h. Since $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0} f_{ave}[x, x+h]$ and since $f(t) \approx f(x)$, we must have $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$.

12. Let $G(x) = \int_0^{\int_0^x \cos(s^2)ds} \cos(t^2)dt$. Find $G'(x)$ [10 pts]

Solution: $G'(x) = \cos\left(\left[\int_0^x \cos(s^2)ds\right]^2\right)\cos(x^2)$

13. Show why $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. [10 pts]

Solution: Let $S = 1 + 2 + \cdots + n$. Then $S = n + (n-1) + \cdots + 1$. Adding the two expressions columnwise we obtain $2S = n(n+1)$. Hence $S = \frac{n(n+1)}{2}$.

14. Suppose that f is an even function with $\int_0^8 f(x)dx = 9$. Evaluate $\int_{-1}^{1} xf(x^2)dx$. [10 pts]

Solution: The function $g(x) = xf(x^2)$ is odd. Hence $\int_{-1}^{1} xf(x^2)dx = 0$